Annular Network with Three Rays and Two Circles and Five Rays and Three Circles

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Abstract:

The graph with three rays and two circles has six different spikes and the graph with five rays and three circles has ten different spikes (edges connected to a boundary vertex), which are altered with initial voltages and currents to compute the conductors γ in the graph using the response matrix Λ. Using the formula $\Lambda = \mathrm{A} - \mathrm{B}\mathrm{D}^{-1}\mathrm{B}^{\mathrm{T}}$, where the blocks correspond to the partitions in the Kirchhoff matrix related to the graphs, the graphs can be recovered by using the value of a given conductor and the different sub-matrices in their respective response matrices, which represent the connections between their boundary vertices. Such connections imply a linear system of equations between voltages and currents, thus, they create a: $Ax = B$ relationship, where *A*, *x*, and *B* are matrices/vectors. Furthermore, the only way to solve the linear system of equations is if there exists a unique connection between the boundary vertices being analyzed.

Introduction:

The purpose of this research is to analyze the annular graphs of rays and circles, more specifically the graph with three rays and two circles and the graph with five rays and three circles. The forward problem stands as: given a set of conductance values for all of the edges in the graph, calculate the response matrix. The inverse problem stated in this paper, which is the main analysis, stands as: given a response matrix and a value for only one conductance, calculate the conductance values for the remaining edges. In contrast, the standard inverse problem would only have given the response matrix. This paper analyses the different families of parameters entered in the response matrix and how the response matrix behaves when the parameters have a certain relationship.

A thorough description of the algorithm to compute the conductance values is given in this paper as well. This step-by-step algorithm is used to recover each and every one of the edges. Some of the theorems involved in this paper will not be discussed in detail; nevertheless, the main concept of each theorem is explained in order to give a better description of each situation that involves each theorem. For further proofs, refer to the bibliography.

The map of conductors into the response matrix, $\gamma \to \Lambda$, is also a major part of this research. In the cases discussed, there will be families of parameters (relationship between edges) that will produce a response matrix with notable characteristics. Some of the algorithms were relatively massive to compute in paper, thus, the larger ones were done in a MATLAB program. There will not be any code provided in this paper, however, all of the cases are very well discussed. As an option, the algorithm may be created in order to verify any operation.

The relations between the entries in the response matrix is also a part of the analysis in this paper.Lastly, the information in this paper may be interpreted in the Electrical Engineering field by the use of currents and voltages; some topics used are:

1. Ohm’s law
2. Kirchhoff’s Current Law
3. Conductance/Resistance

There will be a brief description of each topic when used. More reference can be found in the bibliography, internet, or in any basic-circuits/physics book.

1. The Annular Network with Three Rays and Two Circles

1.1 Recovering Conductances When One Boundary Conductor is Known

When the conductance of a boundary spike is known for $G(3,2)$, the remaining conductances are recoverable. Figure 1.1 shows all the imposed conditions (e.g. 0) and the zero potentials (e.g. 0') that propagate inside the network. In this case of Figure 3.1, $\gamma_{4,10}$ is known to equal a. Setting the current equal to the conductance forces the voltage difference to be one, but since u_4 already equals one, u_{10}becomes zero. Figure 1.2 shows the current flow pattern according to these initial conditions. [2]

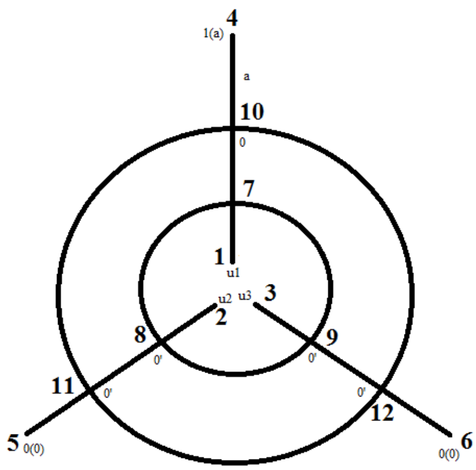


Figure 1.1

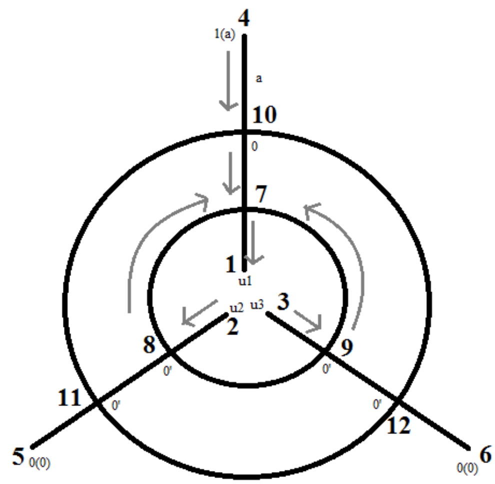


Figure 1.2

Figure 1.1: $G(3,2)$, due to Kirchhoff's current law, the resulting current flow is shown in Figure 1.2.

Figure 1.2: $G(3,2)$ Current Flow. This determines that: $u_2 > 0$, $u_3 > 0$, $u_4 > 0 > u_7 > u_1$.

For simplicity, the conductances were assigned an initial value to analyze the behavior of the response matrix with different a initial values. The following are the initial values for the conductances, starting with the outermost edges to the innermost edges: 1, 1, 1, 2, 2, 2, 3, 3, 3, 4, 4, 4, 5, 5, 5. According to Chapter 3, Section 3.3 (page 33) [1], the resulting Kirchhoff matrix is as shown in Figure 3.3.

In order to recover $G(3,2)$, the voltages u_1, u_2, and u_3 need to be known. Thus, in order to write a linear system of equations to solve for the unknown voltages, the response matrix Λ has to be obtained to calculate those voltages. Moreover, in order to calculate the response matrix from a given Kirchhoff matrix, the Schur Complement has to be taken from the given Kirchhoff matrix according to Theorem 3.9 in Chapter 3, Section 3.5. [1]

$$K = \begin{bmatrix} 5 & 0 & 0 & 0 & 0 & 0 & -5 & 0 & 0 & 0 & 0 & 0 \\ 0 & 5 & 0 & 0 & 0 & 0 & 0 & -5 & 0 & 0 & 0 & 0 \\ 0 & 0 & 5 & 0 & 0 & 0 & 0 & 0 & -5 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & -1 \\ -5 & 0 & 0 & 0 & 0 & 0 & 16 & -4 & -4 & -3 & 0 & 0 \\ 0 & -5 & 0 & 0 & 0 & 0 & -4 & 16 & -4 & 0 & -3 & 0 \\ 0 & 0 & -5 & 0 & 0 & 0 & -4 & -4 & 16 & 0 & 0 & -3 \\ 0 & 0 & 0 & -1 & 0 & 0 & -3 & 0 & 0 & 8 & -2 & -2 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & -3 & 0 & -2 & 8 & -2 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & -3 & -2 & -2 & 8 \end{bmatrix}$$

Figure 1.3: Kirchhoff matrix; this is a symmetric, square matrix, whose sum of its rows and columns is equal to 0.

As given by the formula in page 43, $\Lambda = \mathrm{A} - \mathrm{BD}^{-1}\mathrm{B}^{\mathrm{T}}$, where *K* = $\begin{matrix} A & B \\ B^T & D \end{matrix}$, [1].Suppose a given graph *G* has *n* boundary vertices and *m* interior vertices. The total number of vertices *v* is *v* = *n* + *m* (note: The dimensions of the Kirchhoff matrix are *v x v*). As already mentioned, the Kirchhoff matrix *K* is partitioned in order to get the Schur complement; the upper left hand block *A* represents the boundary vertices. Its dimensions are *n x n,* and it is symmetric. Figure 3.4 shows the partitions of the Kirchhoff matrix:

$$K = \begin{bmatrix} 5 & 0 & 0 & 0 & 0 & 0 & \vdots & -5 & 0 & 0 & 0 & 0 & 0 \\ 0 & 5 & 0 & 0 & 0 & 0 & \vdots & 0 & -5 & 0 & 0 & 0 & 0 \\ 0 & 0 & 5 & 0 & 0 & 0 & \vdots & 0 & 0 & -5 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & \vdots & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & \vdots & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & \vdots & 0 & 0 & 0 & 0 & 0 & -1 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ -5 & 0 & 0 & 0 & 0 & 0 & \vdots & 16 & -4 & -4 & -3 & 0 & 0 \\ 0 & -5 & 0 & 0 & 0 & 0 & \vdots & -4 & 16 & -4 & 0 & -3 & 0 \\ 0 & 0 & -5 & 0 & 0 & 0 & \vdots & -4 & -4 & 16 & 0 & 0 & -3 \\ 0 & 0 & 0 & -1 & 0 & 0 & \vdots & -3 & 0 & 0 & 8 & -2 & -2 \\ 0 & 0 & 0 & 0 & -1 & 0 & \vdots & 0 & -3 & 0 & -2 & 8 & -2 \\ 0 & 0 & 0 & 0 & 0 & -1 & \vdots & 0 & 0 & -3 & -2 & -2 & 8 \end{bmatrix}$$

Figure 1.4: Kirchhoff matrix partitioned into the mentiond blocks from equation

Block A marks the dimensions of blocks B and B^T as well; this is, block B has dimensions *(v – n) x (v – n).* Similarly, block B^Thas the same dimensions. Now that the K matrix has been partitioned into four different blocks, the equation $\Lambda = \mathrm{A} - \mathrm{BD}^{-1}\mathrm{B}^{\mathrm{T}}$ will give the response matrix corresponding to the Kirchhoff matrix as shown in Figure 3.5.

$$\Lambda = (\frac{1}{4393})\begin{bmatrix} 11765 & -4450 & -4450 & -1185 & -840 & -840 \\ -4450 & 11765 & -4450 & -840 & -1185 & -840 \\ -4450 & -4450 & 11765 & -840 & -840 & -1185 \\ -1185 & -840 & -840 & 3577 & -356 & -356 \\ -840 & -1185 & -840 & -356 & 3577 & -356 \\ -840 & -840 & -1185 & -356 & -356 & 3577 \end{bmatrix}$$

Figure 1.5: Response matrix, the entries of this matrix have been expressed as fractions in order to maintain more accurate results, [2].

The main use of the response matrix is to produce a relationship or connection between boundary vertices; moreover, it gives a linear combination of the vertices in which the known voltages are located to the vertices at which the unknown voltages are located. Furthermore, the voltages already known are: $u_4 = 1$, $u_5 = 0$, and $u_6 = 0$, and the unknown are u_1, u_2, and u_3. The linear combination starts at a known voltage (e.g. u_4) and spreads to the unknown voltages; the relationship between a current at vertex *i* (known) due to a voltage at vertex *j* (unknown) is given by the (*i*, *j*) entry in Λ.

$$u_1\lambda_{4,1} + u_2\lambda_{4,2} + u_3\lambda_{4,3} + u_4\lambda_{4,4} = a \tag{1.1}$$

This relationship may be interpreted as: the sum of the (*i*, *j*) entries of Lambda, where each entry corresponds to vertex *i* (whose current and voltage are known) and vertex *j* (whose voltage and current are unknown), times their corresponding voltages at vertex *j* are equal to the current at vertex *i*. These statements lead to:

$$\sum u_i\lambda_{j,i} = I_j \tag{1.2}$$

The corresponding equations to vertices five and six are as follows:

$$u_1\lambda_{5,1} + u_2\lambda_{5,2} + u_3\lambda_{5,3} + u_4\lambda_{5,4} = 0 \tag{1.3}$$

$$u_1\lambda_{6,1} + u_2\lambda_{6,2} + u_3\lambda_{6,3} + u_4\lambda_{6,4} = 0$$

These equations can be written using linear algebra as shown in Figure 1.6:

$$\begin{bmatrix} \lambda_{4,1} & \lambda_{4,2} & \lambda_{4,3} \\ \lambda_{5,1} & \lambda_{5,2} & \lambda_{5,3} \\ \lambda_{6,1} & \lambda_{6,2} & \lambda_{6,3} \end{bmatrix}\begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} a - \lambda_{4,4} \\ -\lambda_{5,4} \\ -\lambda_{6,4} \end{bmatrix}$$

Figure 1.6: the voltage u_4 has been substituted into the equation since it was equal to 1.

The only remaining fact that needs to be taken into consideration in order to solve for the unknown voltages is: the sub-matrix of the response matrix has to be invertible. That is, its determinant has to be different from zero.

According to Theorem 3.13 in Chapter 3, Section 3.7, [1], any set of disjoint vertices P, Q that are connected in only one way through the graph implies the corresponding matrix's determinant is different from zero. Therefore, the only connection between set P = {4, 5, 6} and set Q = {1, 2, 3} is as shown in figure 1.7:

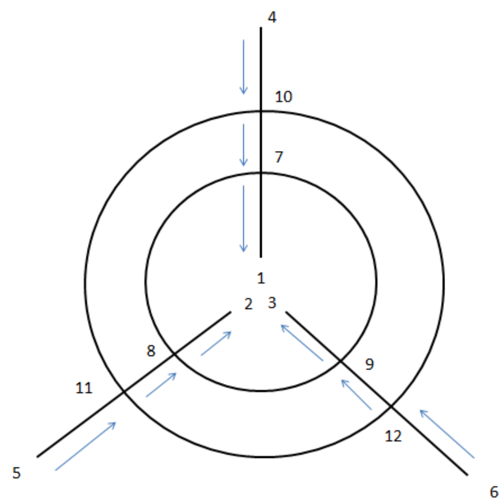


Figure 1.7: Connection between sets P and Q.

By this connection, it is concluded that the matrix in the homogeneous part of the equation is invertible; moreover, it gives a uniquely determined solution to the equation as shown in Figure 1.8:

$$\begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} \lambda_{4,1} & \lambda_{4,2} & \lambda_{4,3} \\ \lambda_{5,1} & \lambda_{5,2} & \lambda_{5,3} \\ \lambda_{6,1} & \lambda_{6,2} & \lambda_{6,3} \end{bmatrix}^{-1} \begin{bmatrix} a - \lambda_{4,4} \\ -\lambda_{5,4} \\ -\lambda_{6,4} \end{bmatrix}$$

Figure 1.8: The solution of the unknown voltages

Let $f = [u_1 \quad u_2 \quad u_3 \quad u_4 \quad u_5 \quad u_6]^T$ be the column vector representing the boundary voltages and let $C = [i_1 \quad i_2 \quad i_3 \quad i_4 \quad i_5 \quad i_6]^T$ be the column vector representing the boundary currents, according to Chapter 3, Section 3.5, [1], the response matrix Λ times the vector *f* gives the corresponding equation for the current, *C*, flowing through the edge adjacent to the boundary voltage (vertex). Figure 1.9 shows such operation.

$$\begin{bmatrix} \lambda_{1,1} & \lambda_{1,2} & \lambda_{1,3} & \lambda_{1,4} & \lambda_{1,5} & \lambda_{1,6} \\ \lambda_{2,1} & \lambda_{2,2} & \lambda_{2,3} & \lambda_{2,4} & \lambda_{2,5} & \lambda_{2,6} \\ \lambda_{3,1} & \lambda_{3,2} & \lambda_{3,3} & \lambda_{3,4} & \lambda_{3,5} & \lambda_{3,6} \\ \lambda_{4,1} & \lambda_{4,2} & \lambda_{4,3} & \lambda_{4,4} & \lambda_{4,5} & \lambda_{4,6} \\ \lambda_{5,1} & \lambda_{5,2} & \lambda_{5,3} & \lambda_{5.4} & \lambda_{5,5} & \lambda_{5,6} \\ \lambda_{6,1} & \lambda_{6,2} & \lambda_{6,3} & \lambda_{6,4} & \lambda_{6,5} & \lambda_{6,6} \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \end{bmatrix} = \begin{bmatrix} i_1 \\ i_2 \\ i_3 \\ i_4 \\ i_5 \\ i_6 \end{bmatrix}$$

Figure 1.9: The characteristics of the response matrix are given in Chapter 5, Section5.1, [1]. This is to emphasize that: $\lambda_{i,j} = \lambda_{j,i}$. The current in the boundary edges is given by this formula.

Using Ohm's law, $i = \frac{\Delta V}{R}$ or $\gamma \Delta V$, where γ is the conductance of the edge ($\frac{1}{R}$), the currents can be calculated. However, the only conductances that can be calculated using the equation above are $\gamma_{2,8}$ and $\gamma_{3,9}$; the reason why is, the voltage drop in these 2 edges is known ($\Delta V = u_i - 0$), in the other hand, the remaining boundary voltages are missing the voltage drop at the end of the edge (e.g. $\Delta V = u_1 - u_7$, u_7 is not known.)

Therefore, the equations for currents i_2 and i_3 are: (1.4)

$$i_2 = u_1\lambda_{2,1} + u_2\lambda_{2,2} + u_3\lambda_{2,3} + \lambda_{2,4}$$

$$i_3 = u_1\lambda_{3,1} + u_2\lambda_{3,2} + u_3\lambda_{3,3} + \lambda_{3,4}$$

Moreover, using Ohm's law, the conductances at the corresponding vertices 2 and 3 are: (1.5)

$$\gamma_{2,8} = \frac{1}{u_2}(u_1\lambda_{2,1} + u_2\lambda_{2,2} + u_3\lambda_{2,3} + \lambda_{2,4})$$

$$\gamma_{3,9} = \frac{1}{u_3}(u_1\lambda_{3,1} + u_2\lambda_{3,2} + u_3\lambda_{3,3} + \lambda_{3,4})$$

The conductances equations can be written explicitly with *a* as a parameter as:

(1.6)

$$\gamma_{2,8} = \gamma_{3,9} = \frac{5(23\text{a} - 21)}{2(14\text{a} - 13)}$$

Note 1.1: These expressions were obtained from a MATLAB program, that simplified the equations with the parameter *a* in them.

Now that the conductances $\gamma_{2,8}$ and $\gamma_{3,9}$ are known, the following step is to change the position of the initial conditions (initial voltages and currents) as shown in Figure 1.10.

Following the first example of initial conditions, the initial voltages are displayed as *V*, initial currents as (*i*), and propagated current and voltage conditions as *v'*. In this case, the zero voltages and zero currents were set at vertices one and two. This is to make use of the conductance $\gamma_{3,9}$ obtained in the previous step (note: the same result can be obtained by using the conductance $\gamma_{2,8}$). A voltage of one was assigned to vertex three (that is ,$u_3 = 1$) and a current equal to the conductance at vertex three. Since the current is equal to five, it makes the voltage drop equal to u_3, however, the voltage at vertex three already equals one, thus, the voltage a vertex nine is equal to zero.

As already mentioned in the first step, the imposed initial conditions propagate currents of zero throughout some of the interior edges; this leads to the current flow pattern shown in Figure 3.3. It also leads to: $u_3 > u_9 > u_{12} > u_6$, $u_5 > 0$, and $u_4 > 0$.

<u>Note 1.2</u>: the values of the voltages obtained in the first step will change throughout the next steps' calculations; however, the values of the voltages and currents obtained in the current step will remain uniquely determined with respect to their corresponding step.

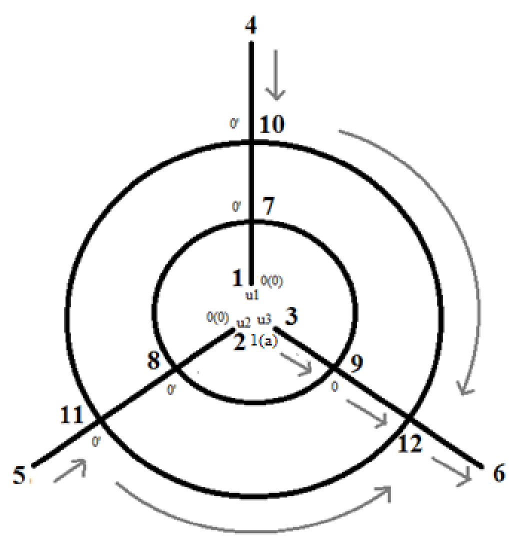


Figure 1.10: *G*(3,2) with the new initial conditions and the pattern of current flow.

Following the same steps from the first example, the connection between the known voltages and the unknown voltages is given by:

$$\begin{bmatrix} \lambda_{1,4} & \lambda_{1,5} & \lambda_{1,6} \\ \lambda_{2,4} & \lambda_{2,5} & \lambda_{2,5} \\ \lambda_{3,4} & \lambda_{3,5} & \lambda_{3,6} \end{bmatrix} \begin{bmatrix} u_4 \\ u_5 \\ u_6 \end{bmatrix} = \begin{bmatrix} -\lambda_{1,3} \\ -\lambda_{2,3} \\ \gamma(3,9) - \lambda_{3,3} \end{bmatrix}$$

Figure 1.11: The solution of the unknown voltages at the second step

The only difference with this connection is the conductance at vertex three expressed as $\gamma_{3,9}$, whereas in the first example, the conductance was expressed as a. Since $\gamma_{3,9}$ is known, it can be plugged into the equation to simplify calculations. Now that voltages in Figure 1.11 are known, the conductances at vertices four and five are the only ones that can be calculated due to the known voltage drop ($\Delta V = u_i - 0$) in their respective edges; using the response matrix, the relation is:

$$\begin{bmatrix} \lambda_{1,1} & \lambda_{1,2} & \lambda_{1,3} & \lambda_{1,4} & \lambda_{1,5} & \lambda_{1,6} \\ \lambda_{2,1} & \lambda_{2,2} & \lambda_{2,3} & \lambda_{2,4} & \lambda_{2,5} & \lambda_{2,6} \\ \lambda_{3,1} & \lambda_{3,2} & \lambda_{3,3} & \lambda_{3,4} & \lambda_{3,5} & \lambda_{3,6} \\ \lambda_{4,1} & \lambda_{4,2} & \lambda_{4,3} & \lambda_{4,4} & \lambda_{4,5} & \lambda_{4,6} \\ \lambda_{5,1} & \lambda_{5,2} & \lambda_{5,3} & \lambda_{5.4} & \lambda_{5,5} & \lambda_{5,6} \\ \lambda_{6,1} & \lambda_{6,2} & \lambda_{6,3} & \lambda_{6,4} & \lambda_{6,5} & \lambda_{6,6} \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \end{bmatrix} = \begin{bmatrix} i_1 \\ i_2 \\ i_3 \\ i_4 \\ i_5 \\ i_6 \end{bmatrix}$$

Figure 1.12: $u_1 = u_2 = 0$, $u_3 = 1$

Thus, the equations for the currents i_4 and i_5 are: (1.7)

$$i_4 = u_4\lambda_{4,4} + u_5\lambda_{5,4} + u_6\lambda_{6,4} + \lambda_{3,4}$$

$$i_5 = u_4\lambda_{4,5} + u_5\lambda_{5,5} + u_6\lambda_{6,5} + \lambda_{3,5}$$

And their respective conductances are: (1.8)

$$\gamma_{4,10} = \frac{1}{u_4}(u_4\lambda_{4,4} + u_5\lambda_{5,4} + u_6\lambda_{6,4} + \lambda_{3,4})$$

$$\gamma_{5,11} = \frac{1}{u_5}(u_4\lambda_{4,5} + u_5\lambda_{5,5} + u_6\lambda_{6,5} + \lambda_{3,5})$$

These conductances equations can be written explicitly with a as a parameter: (1.9)

$$\gamma_{4,10} = \gamma_{5,11} = a$$

With equation 1.9, it is confirmed that the initial value in the first step (conductance of $\gamma_{4,10} = a$) is correct. Moreover, the values of the remaining conductances will depend on a. For simplicity, the initial values (shown in the Kirchhoff matrix) were chosen in a random, easy way so the calculations can be simpler. If a takes the value of one (as shown in the initial values for the simplicity of the Kirchhoff matrix), the edges $\gamma_{3,9}$ and $\gamma_{2,8}$ receive a value of five due to the equation 1.6. This, in fact, proves the value (as assigned in the beginning for the Kirchhoff matrix) of $\gamma_{3,9}$ and $\gamma_{2,8}$ when a is equal to one.

In the first step, the edge $\gamma_{4,10}$ was set to have an initial voltage of one and a current of a, and the edges $\gamma_{5,11}$ and $\gamma_{6,12}$ were set to have voltages and currents of zero in order to solve for the conductances $\gamma_{2,8}$ and $\gamma_{3,9}$. In the second step, the edge $\gamma_{3,9}$ was set to have the initial voltage of one, and the edges $\gamma_{2,8}$ and $\gamma_{1,7}$ were assigned the zero voltages and currents in order to find the conductance at the edges $\gamma_{5,11}$ and $\gamma_{4,10}$.

The process to find the boundary conductances is similar for the edges $\gamma_{1,7}$ and $\gamma_{6,12}$. The initial value of the voltage equal to one is assigned to edges $\gamma_{5,11}$ and $\gamma_{1,7}$, respectively, to obtain the equations for the conductances $\gamma_{1,7}$ and $\gamma_{6,12}$, respectively.

After these calculations, the resultant equations are: (1.10)

$$\gamma_{1,7} = \gamma_{2,8} = \gamma_{3,9} = \frac{5(23\mathrm{a} - 21)}{2(14\mathrm{a} - 13)}$$

$$\gamma_{4,10} = \gamma_{5,11} = \gamma_{6,12} = \mathrm{a}$$

These equations depend on the parameter a, which determines the initial conductance at one of the outermost edges. The remaining edges can be found by using Ohm's law at their corresponding step with their corresponding boundary voltages and conductances.

Referring to Figure 1.2, the voltages u_1, u_2, u_3, and u_4 are known. Also, the currents $i_{2,8}$, $i_{3,9}$, and $i_{4,10}$ are known due to Ohm's law: $\gamma \cdot \Delta V = i$. That is, for example, $\gamma_{2,8} \cdot (u_2 - 0) = i_{2,8}$. Using Kirchhoff's Current Law, which states the currents entering a vertex or node are equal to the currents exiting the vertex or node, $i_{4,10} = i_{10,7}$, $i_{2,8} = i_{8,7}$, $i_{3,9} = i_{9,7}$, and $i_{7,1} = i_{10,7} + i_{8,7} + i_{9,7}$. Due to the zero voltages propagated into the network in Figure 1.1, the current flow pattern is as shown in Figure 1.2.

The only voltage needed to calculate edges $\gamma_{8,7}, \gamma_{9,7}, \gamma_{10,7}$ is u_7. The equation to calculate this voltage, using Ohm's law, is: $(u_7 - u_1)\gamma_{1,7} = i_{7,1}$. Similarly, this process is the same for each step in order to calculate the remaining interior edges with their corresponding boundary voltages. As done before, the equations of the interior edges can be written with a as a parameter in them as: (1.11)

$$\gamma_{8,7} = \gamma_{9,7} = \gamma_{8,9} = \frac{2(23\mathrm{a} - 21)^2}{89\mathrm{a}^2 - 150\mathrm{a} + 63}$$

$$\gamma_{7,10} = \gamma_{8,11} = \gamma_{9,12} = \frac{3\mathrm{a}(23\mathrm{a} - 21)}{89\mathrm{a}^2 - 150\mathrm{a} + 63}$$

$$\gamma_{10,11} = \gamma_{11,12} = \gamma_{10,12} = \frac{4\mathrm{a}^2}{89\mathrm{a}^2 - 150\mathrm{a} + 63}$$

<u>Example 1</u>: let *a* be equal to one, then the corresponding conducttances' values (from outermost to innermost) are: 1, 1, 1, 2, 2, 2, 3, 3, 3, 4, 4, 4, 5, 5, 5. Moreover, let *a* be equal to seven, then the corresponding conductances' values (from outermost to innermost) are: $7, 7, 7, \frac{14}{241}, \frac{14}{241}, \frac{14}{241}, \frac{210}{241}, \frac{210}{241}, \frac{210}{241}, \frac{2800}{241}, \frac{2800}{241}, \frac{2800}{241}, \frac{70}{17}, \frac{70}{17}, \frac{70}{17}$.

At this point, there seems to be a one-to-one relation between the initial, chosen conductances and the response matrix (which comes from the Kirchhoff matrix, whose entries are the chosen conductances). However, if *a* is kept as a parameter in the entries of the Kirchhoff matrix, the resulting response matrix is:

$$\Lambda = (\frac{1}{4393}) \begin{bmatrix} 11765 & -4450 & -4450 & -1185 & -840 & -840 \\ -4450 & 11765 & -4450 & -840 & -1185 & -840 \\ -4450 & -4450 & 11765 & -840 & -840 & -1185 \\ -1185 & -840 & -840 & 3577 & -356 & -356 \\ -840 & -1185 & -840 & -356 & 3577 & -356 \\ -840 & -840 & -1185 & -356 & -356 & 3577 \end{bmatrix}$$

Figure 1.13: The response matrix

This implies the parameter *a* does not play a role in the calculation of the response matrix since it cancels out during the calculations. Thus, for any value of *a*, all of the edges in the graph adjust themselves so the response matrix is kept the same. Moreover, there is not a one-to-one connection between the conductances and the response matrix Λ when the initial conductances have the relationship already mentioned.

<u>Theorem 1.1</u> *The map $\gamma \rightarrow \Lambda$ is infinite to one when the conductances are equal on each layer. Starting with known K and Λ, any choice of a boundary conductor within some range about the original value will lead to a new set of conductors for the network which will then have the same Λ.*

<u>Proof</u>: If the initial conductances are entered in the Kirchhoff matrix as a parameter of a, the resulting response matrix is as shown in Figure 1.13.Thus, since *a* cancels out in the calculation of the response matrix, there are infinitely many choices of *a* that produce the same response matrix Λ. Therefore the map $\gamma \rightarrow \Lambda$ is infinite to one for any choice of a boundary conductor. Furthermore, the graph G(3,2) is not recoverable since its map is not one-to-one.

Despite the fact that the map $\gamma \rightarrow \Lambda$ is infinite to one, there has to be a range of numbers that give the correct set of conductances for any choice *a* in the parameters of the conductances' equations. According to Electrical Engineering, there cannot exist negative resistance, thus, negative conductances are not real. In order to figure out the range of numbers that can be used as a parameter a, each set of conductances must be positive; therefore: (1.12)

1. $a > 0$
2. $\frac{5(23a-21)}{2(14a-13)} > 0$
3. $\frac{2(23a-21)^2}{89a^2-150a+63} > 0$
4. $\frac{3a(23a-21)}{89a^2-150a+63} > 0$
5. $\frac{4a^2}{89a^2-150a+63} > 0$

<u>Note 1.3</u>: Zero resistance implies, in the Mathematics field: an unreal conductance, and in the Electrical Engineering field: a short circuit. That is why these inequalities must be completely positive. In the set of equations 1.12, equation one is positive on $(0, \infty)$, equation two is positive on$(-\infty, \frac{13}{14}) \cup (\frac{13}{14}, \infty)$, and equations three, four, and five are positive on $\left(-\infty, \frac{75-\sqrt{2}}{89}\right) \cup (\frac{75+\sqrt{2}}{89}, \infty)$.

Referring to note 1.3, the range of numbers that can be used as a choice for *a* is: $\left(\frac{13}{14}, \infty\right)$. This concludes that any number in this range given is a valid choice for an initial conductance value. As already mentioned, when one edge changes within a small enough range, the other edges can adjust themselves in order to keep the same response matrix. Also, when the initial set of conductances is equal on each of the five layers, the response matrix always has a special form.

1.2 Characteristics of the Response Matrix

After seeing a certain relationship between the conductances in the layers of the graph *G*(3,2), the response matrix exhibits some notable patterns. There are blocks which have the same value:

$$\Lambda = \frac{1}{\phi}\begin{bmatrix} \Sigma & \alpha & \alpha & \beta & \delta & \delta \\ \alpha & \Sigma & \alpha & \delta & \beta & \delta \\ \alpha & \alpha & \Sigma & \delta & \delta & \beta \\ \beta & \delta & \delta & \Psi & \epsilon & \epsilon \\ \delta & \beta & \delta & \epsilon & \Psi & \epsilon \\ \delta & \delta & \beta & \epsilon & \epsilon & \Psi \end{bmatrix}$$

Figure 1.14: The form of the response matrix when the initial set of conductances is equal on each of the five layers

When one of the boundary conductors is altered and used to compute the other fourteen, they all vary the same amount on each layer without being restricted equal on layers. This has already been shown in example 1. Since each set of conductances has the same equation for their corresponding layer, the value of a conductance remains the same within the layer at which the edge is located at.

<u>Theorem 1.2</u> *Any response matrix which has the form shown in Figure 1.14 and satisfies all sign conditions listed in Chapter 5, Section 5., [1], is a response matrix for G(3,2) if and only if the conductances are equal on each of the five layers.[2].*

<u>Proof</u>: ←"*If the conductances are equal on each of the five layers then the resultant response matrix has the form of Figure 3.14".*

Let the conductances be, from outermost to innermost, $a, a, a, b, b, b, c, c, d, d, d, e, e, e$. The resultant Kirchhoff matrix is shown in Figure 1.15

$$K = \begin{bmatrix} e & 0 & 0 & 0 & 0 & 0 & \vdots & -e & 0 & 0 & 0 & 0 & 0 \\ 0 & e & 0 & 0 & 0 & 0 & \vdots & 0 & -e & 0 & 0 & 0 & 0 \\ 0 & 0 & e & 0 & 0 & 0 & \vdots & 0 & 0 & -e & 0 & 0 & 0 \\ 0 & 0 & 0 & a & 0 & 0 & \vdots & 0 & 0 & 0 & -a & 0 & 0 \\ 0 & 0 & 0 & 0 & a & 0 & \vdots & 0 & 0 & 0 & 0 & -a & 0 \\ 0 & 0 & 0 & 0 & 0 & a & \vdots & 0 & 0 & 0 & 0 & 0 & -a \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ -e & 0 & 0 & 0 & 0 & 0 & \vdots & 2d+e+c & -d & -d & -c & 0 & 0 \\ 0 & -e & 0 & 0 & 0 & 0 & \vdots & -d & 2d+e+c & -d & 0 & -c & 0 \\ 0 & 0 & -e & 0 & 0 & 0 & \vdots & -d & -d & 2d+e+c & 0 & 0 & -c \\ 0 & 0 & 0 & -a & 0 & 0 & \vdots & -c & 0 & 0 & 2b+c+a & -b & -b \\ 0 & 0 & 0 & 0 & -a & 0 & \vdots & 0 & -c & 0 & -b & 2b+c+a & -b \\ 0 & 0 & 0 & 0 & 0 & -a & \vdots & 0 & 0 & -c & -b & -b & 2b+c+a \end{bmatrix}$$

Figure 1.15: Kirchhoff matrix with entries as constant values a, b, c, d, e as given.

Using the equation, $\Lambda = \mathrm{A} - \mathrm{BD}^{-1}\mathrm{B}^{\mathrm{T}}$, the resultant response matrix is:

$$\Lambda = \frac{1}{\phi}\begin{bmatrix} \Sigma & \alpha & \alpha & \beta & \delta & \delta \\ \alpha & \Sigma & \alpha & \delta & \beta & \delta \\ \alpha & \alpha & \Sigma & \delta & \delta & \beta \\ \beta & \delta & \delta & \Psi & \epsilon & \epsilon \\ \delta & \beta & \delta & \epsilon & \Psi & \epsilon \\ \delta & \delta & \beta & \epsilon & \epsilon & \Psi \end{bmatrix}$$

Figure 1.16: Response matrix from Figure 3115

Where: (1.13)

- $\phi = (ac + ae + ce)(ac + 3ad + 3bc + ae + 9bd + 3be + 3cd + ce)$
- $\Sigma = e(a^2c^2 + 3 * abc^2 + 3ac^2d + 3a^2cd + ac^2e + a^2ce + 2bc^2e + 2a^2de + 2c^2de + 9abcd + 3abce + 6abde + 4acde + 6bcde)$
- $\alpha = -e^2(bc^2 + a^2d + c^2d + 3abd + 2acd + 3bcd)$
- $\beta = -ace(ac + ad + bc + ae + 3bd + be + cd + ce)$
- $\delta = -ace(ad + bc + 3bd + be + cd)$
- $\epsilon = -a^2(bc^2 + be^2 + c^2d + 3bcd + 2bce + 3bde)$
- $\Psi = a(c^2e^2 + 2abc^2 + 2abe^2 + 2ac^2d + ace^2 + ac^2e + 3bce^2 + 3bc^2e + 3c^2de + 6abcd + 4abce + 6abde + 3acde + 9bcde)$

Therefore, if the conductances are equal on each layer; the resultant matrix has the form of Figure 1.15.

→*"If the response matrix has the form shown in Figure 1.14 then the conductances are equal on each of the five layers"*

Let the initial conditions be equal to the ones shown in Figure 1.1, then the conductance at vertex four has value of *a* due to the imposed current and voltage drop. Following the same steps and using the same algorithm (shown in step one) and the response matrix shown in Figure 3.15, the conductances in the outermost layer attain the value shown in equation 1.14.

$$
\begin{aligned}
\gamma_{4,10} = \gamma_{5,11} = \gamma_{6,12} = {} & -(ac((ae(bc^2 + c^2d + acd + 3bcd + bce))/(a^2c^2 + a^2e^2 + c^2e^2 + 3abc^2 + 3abe^2 \\
& + 3ac^2d + 3a^2cd + 2ace^2 + 2ac^2e + 2a^2ce + 3bce^2 + 3bc^2e + 3a^2de + 3c^2de \\
& + 9abcd + 6abce + 9abde + 6acde + 9bcde) - (be(a - (a^2(ac^2 + bc^2 + ae^2 \\
& + be^2 + c^2d + ce^2 + c^2e + 3acd + 2ace + 3bcd + 3ade + 2bce + 3bde \\
& + 3cde))/(a^2c^2 + a^2e^2 + c^2e^2 + 3abc^2 + 3abe^2 + 3ac^2d + 3a^2cd + 2ace^2 + 2ac^2e \\
& + 2a^2ce + 3bce^2 + 3bc^2e + 3a^2de + 3c^2de + 9abcd + 6abce + 9abde + 6acde \\
& + 9bcde)))/(ac) - (a(ae + 2be + ce)(bc^2 + be^2 + c^2d + 3bcd + 2bce \\
& + 3bde))/(c(a^2c^2 + a^2e^2 + c^2e^2 + 3abc^2 + 3abe^2 + 3ac^2d + 3a^2cd + 2ace^2 \\
& + 2ac^2e + 2a^2ce + 3bce^2 + 3bc^2e + 3a^2de + 3c^2de + 9abcd + 6abce + 9abde \\
& + 6acde + 9bcde)) + (abe(bc^2 + be^2 + c^2d + 3bcd + 2bce + 3bde))/(c(a^2c^2 \\
& + a^2e^2 + c^2e^2 + 3abc^2 + 3abe^2 + 3ac^2d + 3a^2cd + 2ace^2 + 2ac^2e + 2a^2ce \\
& + 3bce^2 + 3bc^2e + 3a^2de + 3c^2de + 9abcd + 6abce + 9abde + 6acde \\
& + 9bcde))))/(be)
\end{aligned}
\tag{1.14}
$$

Simplifying the equation 1.14, the values for the outermost conductances turn out to simplify to *a*. Due to the massive length of the remaining equations, they will not be included in this paper. Nevertheless, each layer does simplify to its original value of *a*, *b*, *c*, *d*, and *e*. Therefore, if the response matrix has a form of Figure 3.14, the conductances are equal on each of the five layers.

Note 1.4: Theorem 1.2 was proved using a MATLAB program that executed each step of the algorithms used to compute the conductances and matrices already mentioned. It also simplified the massive equations giving a more general form of the equation. In contrast with Theorem 1.1, Theorem 1.2 has a more diverse set of response matrices; however, the relationship is still infinite-to-one since there are infinitely many values for the conductances that produce the same response matrix

Conclusion 1.1: What differentiates both theorems is the relationship in the conductances. In Theorem 1.1, the conductances depend on one parameter, *a;* whereas, in Theorem 1.2, each set of layered conductances is independent from the other sets.

1.3 The Analysis of The $\gamma \to \Lambda$ Map

Up to this point, the conductances on each layer have been equal to each other, whether by numbers, one parameter, or five parameters. Also, their corresponding response matrices have been analyzed with the conclusion of the graph $G(3,2)$ being unrecoverable.

The next step is to analyze if the connection of the map $\gamma \to \Lambda$ is infinite-to-one when all the edges are independent from each other. That is, each conductance has a value of: a, b, c, d, e, f, g, h ... $\in \mathbb{R}$. Unfortunately, the computations are very large to demonstrate there exists a relation between the conductances and the response matrix as done in section 1.1. Nevertheless, there exists another method to prove this condition. In order to prove there exists such set of conductances such that the map $\gamma \to \Lambda$ is always infinite-to-one, the rank theorem will be used to prove there exists at least one free variable in the map L (the image of Λ), given a set of fifteen conductances such that the map $\gamma \to \Lambda$ is infinite-to-one.

Before proving such claim, there are a few conditions (some of them already mentioned in section 1.1) which need to be taken into account:

1. <u>The Function $\gamma \to \Lambda$</u>: Given a set of fifteen conductances in G(3,2), the function that transforms the conductances into the response matrix is $\Lambda = \mathrm{A} - \mathrm{BD}^{-1}\mathrm{B}^{\mathrm{T}}$, where $K = \begin{matrix} A & B \\ B^T & D \end{matrix}$. This transformation goes from $\mathbb{R}^{+15}$ to $\mathbb{R}^{+15}$(where the first fifteen correspond to the edges in the graph and the second fifteen correspond to the strictly, upper-triangular entries in the response matrix). That is, given a set of fifteen conductances, the response matrix equation transforms each γ into a λ. There are fifteen potentially independent entries in the response matrix, these being the upper-triangular entries.

2. <u>The Function $\Lambda \to R$:</u> Let P and Q be sets of three disjoint, boundary vertices in G(3,2), then, there only exists one connection between the set P and the set Q. Suppose $P = \{P_1, P_2, P_3\}$ and $Q = \{Q_1, Q_2, Q_3\}$, and suppose each P_i is connected to each Q_i respectively. Then if P_i is connected to Q_i, the interchange between sets of such P_i and Q_i produces the same one-connection as before. Moreover, $\det(P_1, P_2, P_3;\ Q_1, Q_2, Q_3) = \det\,(P_1, P_2, Q_3;\ Q_1, Q_2, P_3)$, as stated by Chapter 3, Section 3.7, [1]. Furthermore, let $R = \det(P_1, P_2, P_3;\ Q_1, Q_2, Q_3) - \det\,(P_1, P_2, Q_3;\ Q_1, Q_2, P_3)$; since both sets have the same connection in G(3, 2), the determinants are equal to each other, thus: $R = \det(P_1, P_2, P_3;\ Q_1, Q_2, Q_3) - \det(P_1, P_2, Q_3;\ Q_1, Q_2, P_3) = 0$. (e.g. $\det(1,2,4;3,5,6) = \det\,(1,4,5;2,3,6)$ since vertices 2 and 5 are connected, [2]). This transformation goes from $\mathbb{R}^{+15}$ to $\mathbb{R}^{1}$ (where the fifteen potentially-independent entries of the response matrix

are transformed into a real constant). Therefore, R is a function of Λ, which is a function of γ.

3. The Rank Theorem: Let γ be the set of old coordinates that maps to Λ, if the derivative of a map ($D\Lambda$) is not full rank, then there exists a point q in γ where the rank of $D\Lambda$ is maximum . At q, there will be a neighborhood within the coordinates γ that will map each γ to a new set of coordinates X. Let k be the rank of $D\Lambda$ and let n be the number of coordinates in both γ and X, then when new set of coordinates X maps to Λ, the notation of the coordinates X into Λ is: $[x_1, x_2, \dots\ x_n] \to [\lambda_1, \lambda_2, \dots\ \lambda_k, 0_{n-k}]$. That is, the number of γ in the map Λ is equal to the rank of $D\Lambda$, and the remaining entries are zeros.[3].

Theorem 1.3: *If the rank of the map of the response matrix Λ is not full rank then the map $\gamma \to \Lambda$ is infinite-to-one for any given set of conductances.*

Proof: The rank of $D\Lambda$ has to be fourteen or less in order to have at least one parameter that produces infinitely many solutions. To show that $D\Lambda$ has not full rank, there has to be a column vector v such that: $v^{\leftarrow} \cdot D\Lambda = 0$ and $v^{\leftarrow} \neq 0$. This is to demonstrate that $v^{\leftarrow}$ is in the kernel of $D\Lambda$, that is, $D\Lambda$ has not full rank.

Such vector v can be derived from the equation mentioned in condition number two, $R(\Lambda(\gamma))$. Taking the gradient of R, the resultant equation produces a relationship of a vector v and the derivative of the map Λ, that is: $\nabla R\big(\Lambda(\gamma)\big) = \nabla R \cdot D\Lambda$, as stated by the chain rule. As mentioned in condition number two, R = 0, moreover, $\nabla R \cdot D\Lambda = 0$. Since ∇R represents the column vector of the partial derivatives of a certain connection of disjoint boundary vertices in $G(3,2)$ at least one entry in ∇R should be non-zero in order for $D\Lambda$ to have rank of fourteen or less.

Note1.5: Due to the symmetry in the graph, any sets P and Q of disjoint vertices with the properties mentioned in condition number two are equivalent. Thus, for any set P and Q the function R holds. This is to emphasize that any example of P and Q is a valid use of the condition. Also, the dimensions of ∇R are 1x15 (due to the derivatives of fifteen different variables) and the dimensions of $D\Lambda$ are 15x15 (due to the fifteen potentially independent entries in the response matrix)

Let R be the connection of $\mathrm{R} = \det(1,2,4;3,5,6) - \det(1,4,5;2,3,6) = 0$. As already mentioned in note 1.5, any connection for P and Q with the characteristics of condition two, holds in R. Since there only exists one way to connect such sets ($1 \to 3, 4 \to 6, 2 \to 5$) both determinants are equal to each other. Let Λ_1 be the first set of connections and Λ_2 be the second set of connections, Figure 1.17 shows the matrices.

$$\Lambda_1 = \begin{vmatrix} \lambda_{1,3} & \lambda_{1,5} & \lambda_{1,6} \\ \lambda_{2,3} & \lambda_{2,5} & \lambda_{2,6} \\ \lambda_{4,3} & \lambda_{4,5} & \lambda_{4,6} \end{vmatrix}, \Lambda_2 = \begin{vmatrix} \lambda_{1,2} & \lambda_{1,3} & \lambda_{1,6} \\ \lambda_{4,2} & \lambda_{4,3} & \lambda_{4,6} \\ \lambda_{5,2} & \lambda_{5,3} & \lambda_{5,6} \end{vmatrix}$$

Figure 1.17: Matrices of the two different connections

Note 1.6: As mentioned in Chapter 3, Section 3.7 [1], the connection given by such boundary vertices is given by the determinants in Figure 1.17.

The gradient of R involves taking partial derivatives of every single λ in the equation:

$$R = \begin{vmatrix} \lambda_{1,3} & \lambda_{1,5} & \lambda_{1,6} \\ \lambda_{2,3} & \lambda_{2,5} & \lambda_{2,6} \\ \lambda_{4,3} & \lambda_{4,5} & \lambda_{4,6} \end{vmatrix} - \begin{vmatrix} \lambda_{1,2} & \lambda_{1,3} & \lambda_{1,6} \\ \lambda_{4,2} & \lambda_{4,3} & \lambda_{4,6} \\ \lambda_{5,2} & \lambda_{5,3} & \lambda_{5,6} \end{vmatrix}$$

Figure 1.18: The function R in terms of the matrices connections

Since there are only thirteen λ terms, the column vector of ∇R already contains two zeros due to the partial derivatives of the λ terms missing in the equation R. The following are the partial derivatives of R with respect to each λ: (1.15)

- $\partial\lambda_{1,2} = -\begin{vmatrix} \lambda_{4,3} & \lambda_{4,6} \\ \lambda_{5,3} & \lambda_{5,6} \end{vmatrix}$
- $\partial\lambda_{1,3} = \begin{vmatrix} \lambda_{2,5} & \lambda_{2,6} \\ \lambda_{4,5} & \lambda_{4,6} \end{vmatrix} + \begin{vmatrix} \lambda_{4,2} & \lambda_{4,6} \\ \lambda_{5,2} & \lambda_{5,6} \end{vmatrix}$
- $\partial\lambda_{1,4} = 0$
- $\partial\lambda_{1,5} = -\begin{vmatrix} \lambda_{2,3} & \lambda_{2,6} \\ \lambda_{4,5} & \lambda_{4,6} \end{vmatrix}$
- $\partial\lambda_{1,6} = \begin{vmatrix} \lambda_{2,3} & \lambda_{2,5} \\ \lambda_{4,3} & \lambda_{4,5} \end{vmatrix} - \begin{vmatrix} \lambda_{2,4} & \lambda_{3,4} \\ \lambda_{2,5} & \lambda_{3,5} \end{vmatrix}$
- $\partial\lambda_{2,3} = -\begin{vmatrix} \lambda_{1,5} & \lambda_{1,6} \\ \lambda_{4,5} & \lambda_{4,6} \end{vmatrix}$
- $\partial\lambda_{2,4} = \begin{vmatrix} \lambda_{1,3} & \lambda_{1,6} \\ \lambda_{5,3} & \lambda_{5,6} \end{vmatrix}$
- $\partial\lambda_{2,5} = \begin{vmatrix} \lambda_{1,3} & \lambda_{1,6} \\ \lambda_{4,3} & \lambda_{4,6} \end{vmatrix} + \begin{vmatrix} \lambda_{1,3} & \lambda_{1,6} \\ \lambda_{4,3} & \lambda_{4,6} \end{vmatrix}$
- $\partial\lambda_{2,6} = -\begin{vmatrix} \lambda_{1,3} & \lambda_{1,5} \\ \lambda_{4,3} & \lambda_{4,5} \end{vmatrix}$
- $\partial\lambda_{3,4} = -(\begin{vmatrix} \lambda_{1,5} & \lambda_{1,6} \\ \lambda_{2,5} & \lambda_{2,6} \end{vmatrix} + \begin{vmatrix} \lambda_{1,2} & \lambda_{1,6} \\ \lambda_{5,2} & \lambda_{5,6} \end{vmatrix})$

- $\partial\lambda_{3,5} = -\begin{vmatrix}\lambda_{1,2} & \lambda_{1,6}\\ \lambda_{4,2} & \lambda_{4,6}\end{vmatrix}$
- $\partial\lambda_{3,6} = 0$
- $\partial\lambda_{4,5} = -\begin{vmatrix}\lambda_{1,3} & \lambda_{1,6}\\ \lambda_{2,3} & \lambda_{2,6}\end{vmatrix}$
- $\partial\lambda_{4,6} = \begin{vmatrix}\lambda_{1,2} & \lambda_{1,3}\\ \lambda_{5,2} & \lambda_{5,3}\end{vmatrix} - \begin{vmatrix}\lambda_{1,3} & \lambda_{1,5}\\ \lambda_{2,3} & \lambda_{2,5}\end{vmatrix}$
- $\partial\lambda_{5,6} = -\begin{vmatrix}\lambda_{1,2} & \lambda_{1,3}\\ \lambda_{4,2} & \lambda_{4,3}\end{vmatrix}$

$$\nabla R = [\partial\lambda_{1,2}, \partial\lambda_{1,3}, \partial\lambda_{1,4}, \partial\lambda_{1,5}, \partial\lambda_{1,6}, \partial\lambda_{2,3}, \partial\lambda_{2,4}, \partial\lambda_{2,5}, \partial\lambda_{2,6}, \partial\lambda_{3,4}, \partial\lambda_{3,5}, \partial\lambda_{3,6}, \partial\lambda_{4,5}, \partial\lambda_{4,6}, \partial\lambda_{5,6}]^T$$

Remark 1.1: Entry $\lambda_{i,j} = \lambda_{j,i}$, ∇R has fifteen variables, which are the partial derivatives of the upper-triangle entries on Λ representing the connections between the boundary vertices.

In order for ∇R to be in the kernel of $D\Lambda$, at least one entry in $\nabla\Lambda$ must not be zero. To prove at least one entry is non-zero, any matrix's determinant, from equations 1.15 must be non-zero. For analysis, let $\partial\lambda_{2,4} = \partial\lambda_{4,5} = 0$. That is:

$$\begin{vmatrix}\lambda_{1,3} & \lambda_{1,6}\\ \lambda_{5,3} & \lambda_{5,6}\end{vmatrix} = \begin{vmatrix}\lambda_{1,3} & \lambda_{1,6}\\ \lambda_{2,3} & \lambda_{2,6}\end{vmatrix} = 0.$$

Figure 1.19: The assumption of determinants are equal to zero

The negative sign of $\partial\lambda_{4,5}$ is eliminated due to the zero condition. If both determinants are equal to zero then their respective first rows are proportionally equivalent to their second rows, that is: $\lambda_{1,3} = C \cdot \lambda_{5,3} = D \cdot \lambda_{2,3}$ and $\lambda_{1,6} = C \cdot \lambda_{5,6} = D \cdot \lambda_{2,6}$ for some $C, D \in \mathbb{R}$. Moreover, if the second row of both determinants is proportionally equivalent to their first row and their first row is the same for both determinants then the second row of the first determinant is proportionally equivalent to the second row of the second determinant. That is:

(1.16)

$$\lambda_{5,3} = \frac{D}{C} \cdot \lambda_{2,3} \text{ and } \lambda_{5,6} = \frac{D}{C} \cdot \lambda_{2,6} \text{ for some } C, D \in \mathbb{R}$$

Note 1.7: Since these equations represent entries in the response matrix of the graph G(3,2), the values for C and D are non-zero.

These entries can be written in matrix form as:

$$\begin{vmatrix} \lambda_{5,3} & \lambda_{5,6} \\ \frac{D}{C}\lambda_{5,3} & \frac{D}{C}\lambda_{5,6} \end{vmatrix} = \begin{vmatrix} \lambda_{5,3} & \lambda_{5,6} \\ \lambda_{2,3} & \lambda_{2,6} \end{vmatrix}$$

Figure 1.20: The relationship of the equations 1.16

Since the rows of Figure 1.20 are proportionally equivalent, its determinant is also zero. That is:

$$\begin{vmatrix} \lambda_{5,3} & \lambda_{5,6} \\ \lambda_{2,3} & \lambda_{2,6} \end{vmatrix} = 0$$

Figure 1.21: The new connection due to the assumptions made in Figures 1.20 and 1.19

The matrix in Figure 1.21 represents the connection of vertices two and five to vertices three and six. By lemma 3.12 in Chapter 3, Section 3.7, [1], if there is a unique connection or an even set of permutations between boundary vertices in a connected resistor network, then the determinant corresponding to the connection is non-zero.

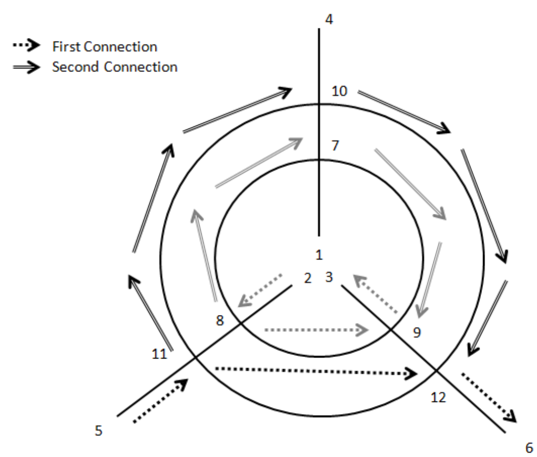
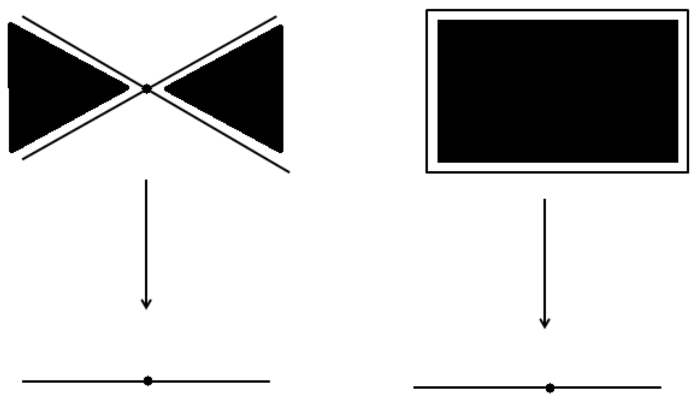


Figure 1.22: The only way to connect vertices two and five to three and six

Since the only way to connect the vertices is: $2 \rightarrow 3, 5 \rightarrow 6$, (as shown in Figure 1.22) then, by contradiction, the determinant in Figure 1.21 is non-zero. Moreover, the relationship established in Figure 1.20 is false, that is, the second rows of each determinant in Figure 1.19 are not proportionally equivalent. Furthermore, the assumption made in Figure 1.19 is false. Therefore, $\partial\lambda_{2,4} \neq 0$ and $\partial\lambda_{4,5} \neq 0$.

<u>Conclusion 1.2:</u> Since at least $\partial\lambda_{2,4} \neq 0$ and $\partial\lambda_{4,5} \neq 0$, the entries of ∇R are not all zero. Thus, ∇R is in the kernel of $D\Lambda$ and it does not have full rank.

However, if the matrix $D\Lambda$ has rank different from fourteen and still is not full rank there may be some issues. Suppose each γ is being looked as a variable in a 15-dimensional space. Then if there are two or more independent variables (these come from the matrix $D\Lambda$ not having full rank) and those variables are fixed, such that the map from $\gamma \to \Lambda$ has an image of the intersection of however many independent variables there are, then the relationship might become one-to-one at some point.



(A) (B)

Figure 1.23: Figure (A) shows the intersection, in this case, of two planes that produce an infinite-to-one relationship everywhere except at some point (middle). This might happen when there are more than two independent variables. Figure (B) shows the plane of just one independent variable, that is, only one variable is fixed, thus, it produces always an infinite-to-one relationship.

Lemma 1.1: *If the rank of $D\Lambda$ is fourteen then for each Λ there are infinitely many conductances γ such that* $\Lambda = \Lambda_{\gamma}$.

Proof: Before proving such claim, there are a few remarks that need to be taken into consideration:

1. Given any set of conductances which transform into a response matrix, $(\gamma_1 \ldots \gamma_{15}) \to \Lambda$, let the top edge be fixed; that is, let $\gamma_{15} = a$. (This is the same assumption made in Figure 1.1, for simplicity let $\gamma_{15} = \gamma_{4,10}$.)
2. Using Λ and a, the original conductances can be obtained back (The same process done in the first Case in Section 1.1)

3. According to the initial conductances, each edge can be expressed as a ratio of two polynomials (in Section 1.1, the initial conductances were equal on each layer, however, if they are entered with different values each edge will ultimately depend on the initial parameter a). Let S be the

function of such polynomials depending on the parameter *a*, that is:

$$S(\lambda_{i,j}a) = \frac{P(\lambda_{i,j})(a)}{Q(\lambda_{i,j})(a)} \qquad (1.17)$$

thus, function S is defined in a open set in $\mathbb{R}^{15}$ which contains unique entries of the response matrix Λ.

4. Figure 1.24 shows the relationship of the γ space when transformed into Λ space, which then, together with parameter, *a*, creates the function S. Function S then transforms back to γ space.

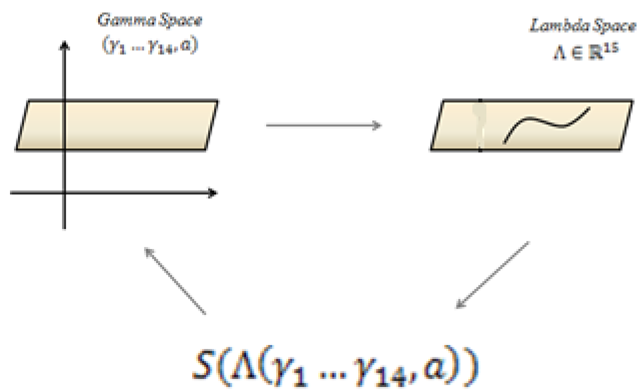


Figure 1.24: When one conductance, *a*, is fixed, the image of the conductances into the Lambda space is given by the response matrix Λ. Such image is a subset of the Lambda space, which is, the set of response matrices corresponding to the conductances entered in the Gamma space. The function S, which is composed of the entries in the response matrix, is defined in an open neighborhood around the image in the Lambda space; this function transforms back each $\lambda_{i,j}$into the Gamma space.

Since the function S transforms every $\lambda_{i,j}$ in the Lambda space back to the Gamma space. That is: (1.18)

$$S \circ \Lambda(\gamma_1 \dots \gamma_{14}, a) = (\gamma_1 \dots \gamma_{14}, a) \in \mathbb{R}^{15}$$

Let $F = S \circ \Lambda = (\gamma_1 \dots \gamma_{14}, a)$, then, the matrix containing the partial derivatives with respect to the variables γ, $\frac{\partial F}{\partial \gamma}$, is:

$$\frac{\partial F}{\partial \gamma} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

Figure 1.25: The matrix of the derivatives with respect to the variables γ. It has rank fourteen.

Conclusion 1.3: Then, $DF = DS \circ D\Lambda = \frac{\partial F}{\partial \gamma}$. Since $\frac{\partial F}{\partial \gamma}$represents the matrix in Figure 1.25 with rank fourteen, the left hand side of the equation $DS \circ D\Lambda$ has to have rank of at least fourteen in order for it to not be in the kernel of $\frac{\partial F}{\partial \gamma}$. Moreover, $D\Lambda$ has to have rank of at least fourteen.

Referring back to conclusion 1.2, $D\Lambda$ is not full rank (Rank $(D\Lambda) < 15$). Then, according to conclusion 1.3, the rank of $D\Lambda$ must be at least fourteen (Rank $(D\Lambda) \geq 14$). Therefore, the rank of $D\Lambda$ is strictly fourteen.

Therefore, referring to Lemma 1.1, for any given response matrix corresponding to the graph G(3,2), there are infinitely many sets of conductances that map to the same response matrix. Moreover, any G(3,2) graph will be unrecoverable due to the infinite-to-one relationship between the conductances of the graph and its response matrix.

2 The Annular Network with Five Rays and Three Circles

2.1 Recovering Conductances When One Boundary Conductor is Known

In order to compute the conductance in the edges of this larger network, the algorithm used in *G*(3,2) will be used in the exact same way only for the conductance in the boundary edges (spikes) due to the increase in the number edges from *G*(3,2) to *G*(5,3).

The initial conditions for the network are shown in the Figure 2.1; the conductor at vertex six has a current equal to the value of its conductance, thus, forcing the voltage drop to be equal to one. Due to the zero currents and zero voltages in the network, the conditions shown in Figure 2.1 show the propagated zero voltages into the network due to Kirchhoff's Current Law as well. Thus, the resulting current flow is also shown in Figure 2.2.

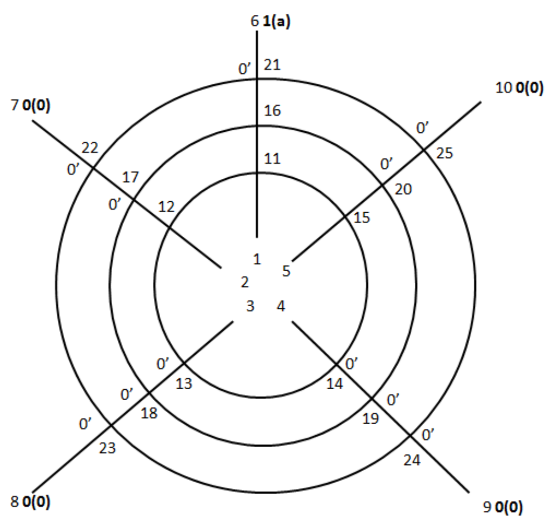


Figure 2.1

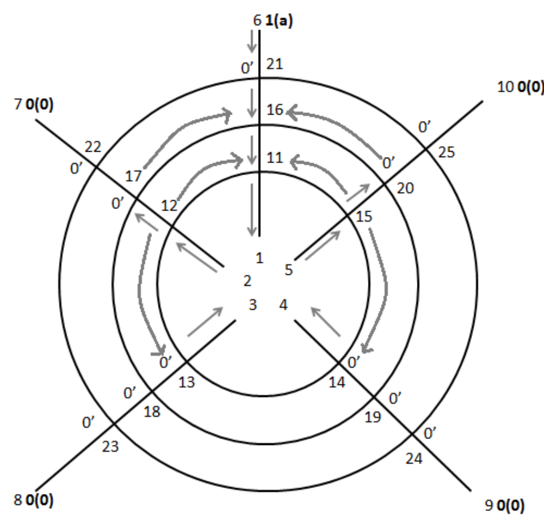


Figure 2.2

<u>Note 2.1:</u> The zero voltages propagated in Figure 2.1 are obtained from Kirchhoff's Current Law. Also, due to the current flow in Figure 2.2, the voltages at each node attain a certain value: $u_6 > u_{21} > u_{16} > u_{11} > u_1, u_2 > u_{12} > u_{17}, u_5 > u_{15} > u_{20}$.

One aspect that will be different from *G*(3,2), in *G*(5,3) will be the initial response matrix. In *G*(3,2) some conductance values were arbitrarily assigned to the edges in order to notice the characteristics of the edges in terms of the initial conductance value a.

Alternatively, in *G*(5,3), there will be no initial conductance values to the edges. This is to provide a more general algorithm to compute all of the edges in the graph. As already seen in *G*(3,2), given a response matrix Λ and an initial conductance value *a*, all of the remaining edges will be a function of the parameter a. Giving initial values to the edges will produce just an example (as already seen in 1.1) of a certain set of conductances, whereas, this algorithm will give a more general form of working through the graph for any given response matrix and initial conductance.

The system of linear equations obtained from the known voltages to the unknown voltages is as given:

$$\begin{bmatrix} \lambda_{6,1} & \lambda_{6,2} & \lambda_{6,3} & \lambda_{6,4} & \lambda_{6,5} \\ \lambda_{7,1} & \lambda_{7,2} & \lambda_{7,3} & \lambda_{7,4} & \lambda_{7,5} \\ \lambda_{8,1} & \lambda_{8,2} & \lambda_{8,3} & \lambda_{8,4} & \lambda_{8,5} \\ \lambda_{9,1} & \lambda_{9,2} & \lambda_{9,3} & \lambda_{9,4} & \lambda_{9,5} \\ \lambda_{10,1} & \lambda_{10,2} & \lambda_{10,3} & \lambda_{10,4} & \lambda_{10,5} \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \end{bmatrix} = \begin{bmatrix} a - \lambda_{6,6} \\ -\lambda_{7,6} \\ -\lambda_{8,6} \\ -\lambda_{9,6} \\ -\lambda_{10,6} \end{bmatrix}$$

Figure 2.3: The system of linear equations involving the unknown voltages and the known voltages. These equations follow the same example mentioned in Section 1.1.

Since there is only one connection between the exterior spikes and the interior spikes, the sub-matrix of the response matrix is invertible (determinant is non-zero). Moreover, there exists a unique solution for the voltages: u_1, u_2, u_3, u_4, u_5.

Due to the voltage drop known ($\Delta V = u_i - 0$) in the edges $\gamma_{3,13}$ and $\gamma_{4,14}$, the corresponding conductance can be calculated by using Ohm's law. The currents at the boundary edges are given by the following linear system of equations shown in Figure 2.4.

$$\begin{bmatrix} \lambda_{1,1} & \lambda_{1,2} & \lambda_{1,3} & \lambda_{1,4} & \lambda_{1,5} & \lambda_{1,6} & \lambda_{1,7} & \lambda_{1,8} & \lambda_{1,9} & \lambda_{1,10} \\ \lambda_{2,1} & \lambda_{2,2} & \lambda_{2,3} & \lambda_{2,4} & \lambda_{2,5} & \lambda_{2,6} & \lambda_{2,7} & \lambda_{2,8} & \lambda_{2,9} & \lambda_{2,10} \\ \lambda_{3,1} & \lambda_{3,2} & \lambda_{3,3} & \lambda_{3,4} & \lambda_{3,5} & \lambda_{3,6} & \lambda_{3,7} & \lambda_{3,8} & \lambda_{3,9} & \lambda_{3,10} \\ \lambda_{4,1} & \lambda_{4,2} & \lambda_{4,3} & \lambda_{4,4} & \lambda_{4,5} & \lambda_{4,6} & \lambda_{4,7} & \lambda_{4,8} & \lambda_{4,9} & \lambda_{4,10} \\ \lambda_{5,1} & \lambda_{5,2} & \lambda_{5,3} & \lambda_{5,4} & \lambda_{5,5} & \lambda_{5,6} & \lambda_{5,7} & \lambda_{5,8} & \lambda_{5,9} & \lambda_{5,10} \\ \lambda_{6,1} & \lambda_{6,2} & \lambda_{6,3} & \lambda_{6,4} & \lambda_{6,5} & \lambda_{6,6} & \lambda_{6,7} & \lambda_{6,8} & \lambda_{6,9} & \lambda_{6,10} \\ \lambda_{7,1} & \lambda_{7,2} & \lambda_{7,3} & \lambda_{7,4} & \lambda_{7,5} & \lambda_{7,6} & \lambda_{7,7} & \lambda_{7,8} & \lambda_{7,9} & \lambda_{7,10} \\ \lambda_{8,1} & \lambda_{8,2} & \lambda_{8,3} & \lambda_{8,4} & \lambda_{8,5} & \lambda_{8,6} & \lambda_{8,7} & \lambda_{8,8} & \lambda_{8,9} & \lambda_{8,10} \\ \lambda_{9,1} & \lambda_{9,2} & \lambda_{9,3} & \lambda_{9,4} & \lambda_{9,5} & \lambda_{9,6} & \lambda_{9,7} & \lambda_{9,8} & \lambda_{9,9} & \lambda_{9,10} \\ \lambda_{10,1} & \lambda_{10,2} & \lambda_{10,3} & \lambda_{10,4} & \lambda_{10,5} & \lambda_{10,6} & \lambda_{10,7} & \lambda_{10,8} & \lambda_{10,9} & \lambda_{10,10} \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \\ u_7 \\ u_8 \\ u_9 \\ u_{10} \end{bmatrix} = \begin{bmatrix} i_1 \\ i_2 \\ i_3 \\ i_4 \\ i_5 \\ i_6 \\ i_7 \\ i_8 \\ i_9 \\ i_{10} \end{bmatrix}$$

Figure 2.4: The linear system of equations when the response matrix is multiplied times the boundary voltages to compute the boundary currents

Therefore, the currents at vertices three and four are: (2.1)

$$i_3 = u_1\lambda_{3,1} + u_2\lambda_{3,2} + u_3\lambda_{3,3} + u_4\lambda_{3,4} + u_5\lambda_{3,5} + u_6\lambda_{3,6} + u_7\lambda_{3,7} + u_8\lambda_{3,8} + u_9\lambda_{3,9} + u_{10}\lambda_{3,10}$$

$$i_4 = u_1\lambda_{4,1} + u_2\lambda_{4,2} + u_3\lambda_{4,3} + u_4\lambda_{4,4} + u_5\lambda_{4,5} + u_6\lambda_{4,6} + u_7\lambda_{4,7} + u_8\lambda_{4,8} + u_9\lambda_{4,9} + u_{10}\lambda_{4,10}$$

And their corresponding conductance is: (2.2)

$$\gamma_{3,13} = \frac{i_3}{u_3}$$

$$\gamma_{4,14} = \frac{i_4}{u_4}$$

Following the same procedure as $G(3,2)$, now that a pair of the conductances in the interior spikes are known, they can be used to compute a pair of conductances in the exterior spikes. This is called “rotating the picture”, which is, changing the initial conditions and making use of the new conductances computed.

Note 2.2: the initial current at the vertex having the voltage of one must have the current equal to its conductance in order to make the voltage drop equal to one.

After computing all of the conductances on the boundary edges, the remaining edges are only the interior edges. In order to compute the values for the interior edges, Ohm’s law and the linear system of equations in Figure 2.4 (this is to use the boundary currents) will be used. Referring to Figure 2.2, the conductances, as well as the voltages at the interior spikes, are already known.

Given the current at edges $\gamma_{2,12}, \gamma_{3,13}, \gamma_{4,14}$, the voltage drop can be computed from using Ohm's law (e.g. $(u_2 - u_{12})\gamma_{2,12} = i_2$). Following this example, the voltages u_{11}, u_{12}, u_{15} can be computed. Due to Kirchhoff's current law, the current $i_{12,13}$ is equal to the current $i_{13,3}$ (Figure 2.2), similarly, current $i_{15,14}$ is equal to $i_{14,4}$. The corresponding edges to those currents, $\gamma_{12,13}, \gamma_{15,14}$, can be computed using Ohm's law, that is: (2.3)

$$\gamma_{12,13} = \frac{i_{12,13}}{u_{12} - u_{13}}$$

$$\gamma_{15,14} = \frac{i_{15,14}}{u_{15} - u_{14}}$$

By rotating the picture (changing the initial conditions in the exterior spikes), the interior ring, as shown in Figure 2.5, can be computed; that is, the conductors that make up the interior ring are all computed.

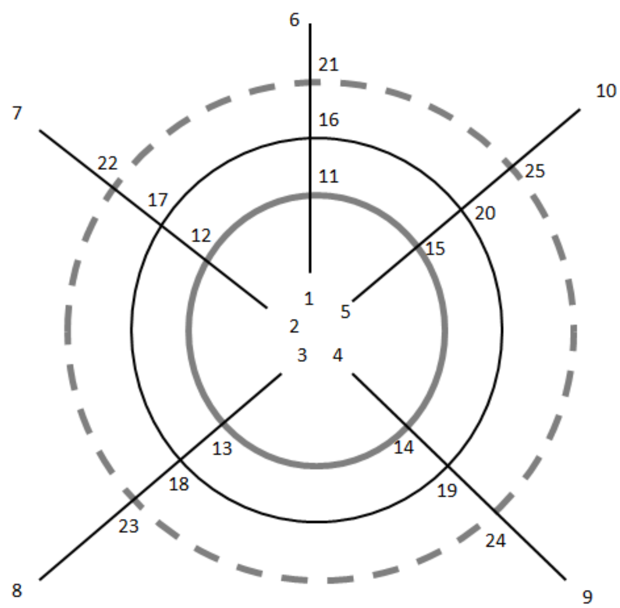


Figure 2.5: The graph G(5,3) with the interior ring is displayed in a thicker line with a different color and the exterior ring is displayed with a dashed line.

Following the same approach when the boundary conductances (spikes) were computed, if the initial conditions are assigned in the exterior spikes, the conductances obtained will be in the interior spikes and vice versa. Similarly in computing the conductances in the rings, if the initial conditions are assigned in the exterior spikes the interior ring's conductances can be computed using Ohm's law. Therefore, if the initial conditions are assigned in the interior spikes the exterior ring's conductances can be computed.

Let the conductors, located in-between the middle and the exterior rings and the middle and the interior rings, be called "bridges". Referring to Figure 2.2, two bridges can be computed, those are: $\gamma_{12.17}, \gamma_{15,20}$. Since the conductances are known in the interior ring as well as in the interior spikes and their respective voltages, their respective currents are also known. Moreover, using Kirchhoff's Current Law, the current flowing through the bridges is also known: (2.4)

$$i_{12,17} = i_{2,12} - i_{12,11} - i_{12,13}$$

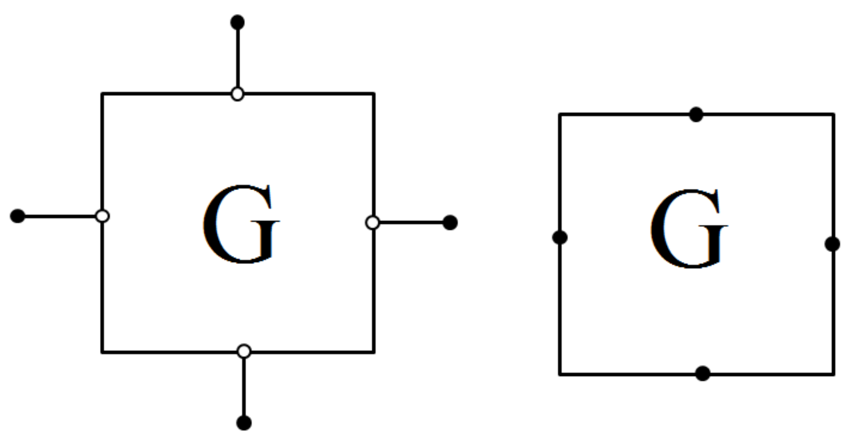
$$i_{15,20} = i_{5,15} - i_{15,11} - i_{15,14}$$

Due to known voltage drop in the edges $\gamma_{12.17}, \gamma_{15,20}$, (e.g. $\Delta V = u_{12} - u_{17}$, $u_{17} = 0$), their respective conductance can be computed. Similarly, by rotating the graph, the remaining bridges can be computed. The bridges adjacent to the exterior spikes can be computed by having the initial conditions assigned in the interior spikes.

2.2 Contracting the Graph $G(5,3)$ to Recover the Conductances When One Conductor is Known

There exists another method to compute the conductances in the graph with five rays and three circles. This method may be more convenient for bigger graphs since it reduces the number of boundary edges. Nevertheless, it still relates to the procedure mentioned in section 2.1; in fact, the algorithm to compute the boundary edges (spikes) in this section is the exact same algorithm used in section 2.1. In Chapter 6, Section 6.3 [1], there is a description/algorithm on how to create a boundary spike in a given graph G. The approach, however, used in this paper will be opposite to the one shown in Section 6.3. Section 6.3 talks about creating a boundary spike, whereas, this method will be about contracting or "killing" a boundary spike.

Suppose a graph *G* has *n* boundary vertices, moreover, it has *n* boundary spikes. Such graph has already the conductances for its spikes computed; when the spikes get contracted, the interior vertices that were adjacent to the boundary vertices whose spike was contracted will become the new boundary vertices. If the new boundary vertices had a connection between themselves before contracting the spikes, after contracting the spikes, the edges joining the two new boundary vertices will be a boundary-to-boundary edge.



(A) (B)

Figure 2.6: Figure A shows the graph G with, in this case, with four boundary spikes and four interior nodes connected to the four boundary vertices in the spikes. After being contracted, the interior nodes become boundary vertices.

Suppose each spike has conductance of $\xi_1, \xi_2, \ldots, \xi_n$ and the response matrix Λ was used to compute such conductances for the spikes in the graph. In order to contract each spike, a conductance of $-\xi_1, -\xi_2, \ldots, -\xi_n$ will be added to each spike respectively as shown in Figure 2.7.

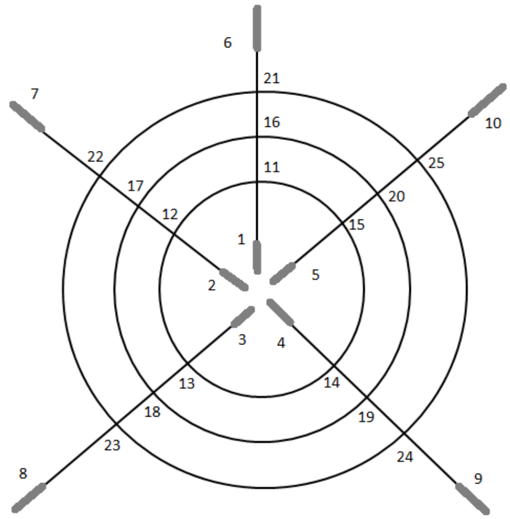


Figure 2.7: *G(*5,3) with the negative conductances added to the spikes, displayed in a thicker, grey line.

By this method, the spikes, whether they are interior or exterior, will be contracted and the resulting graph will not have any spikes. However, the response matrix Λ will be changed. Let Φ be the new Kirchhoff matrix for the contracted graph, where $\Phi = \begin{bmatrix} \mathrm{A} & \mathrm{B} \\ \mathrm{B}^{\mathrm{T}} & \mathrm{D} \end{bmatrix}$, such partitions of the matrix are: (2.5)

$$A = \begin{bmatrix} -\xi_1 & \ldots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \ldots & -\xi_n \end{bmatrix}$$

$$B = B^T = \begin{bmatrix} \xi_1 & \ldots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \ldots & \xi_n \end{bmatrix}$$

$$D = \begin{bmatrix} \lambda_{1,1} - \xi_1 & \ldots & \lambda_{1,n} \\ \vdots & \ddots & \vdots \\ \lambda_{n,1} & \ldots & \lambda_{n,n} - \xi_n \end{bmatrix}$$

<u>Note 2.3</u>: A is the diagonal matrix with the added, negative conductances in their respective spots, B and its transpose are the negative of matrix A, and D is sum of the response matrix Λ and matrix A.

The new response matrix Λ' is given by: $\Lambda' = A - BD^{-1}B^T$. This matrix will be used in the same way as the one seen in the graphs with the spikes on them. The new graph *G*', after having its spikes contracted, will look as shown in Figure 2.8.

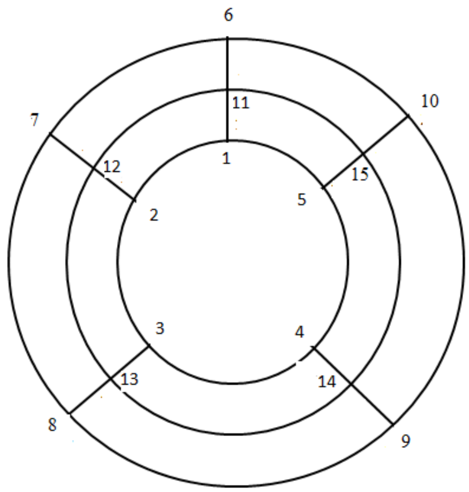


Figure 2.8: G(5,3) with the contracted spikes

<u>Note 2.4</u>: Since the contracted *G*(5,3) still has ten boundary vertices, its response matrix Λ', is 10 x 10.

In order to compute the conductances in the graph, initial conditions will be assigned to boundary vertices, as before. Such conditions will be assigned to the outer-boundary vertices. As already seen in *G*(3,2) and Section 2.1, the algorithm remains the same as far as rotating the picture and assigning the initial conditions to the inner-boundary vertices. Figure 2.9 shows the voltages propagated into the network due to the initial conditions as well as the current flow.

The equations to solve for the inner-boundary vertices' voltages are given in Figure 2.10. The corresponding currents to the inner-boundary vertices are given in equation 2.11. Lastly, the equation to compute the conductances given in Figure 2.9 due to the corresponding initial conditions is given in equation 2.6.

<u>Note 2.5</u>: All of these procedures have the same approach as Section 2.1, that is, to compute the remaining conductances by rotating the picture and changing the initial conditions.

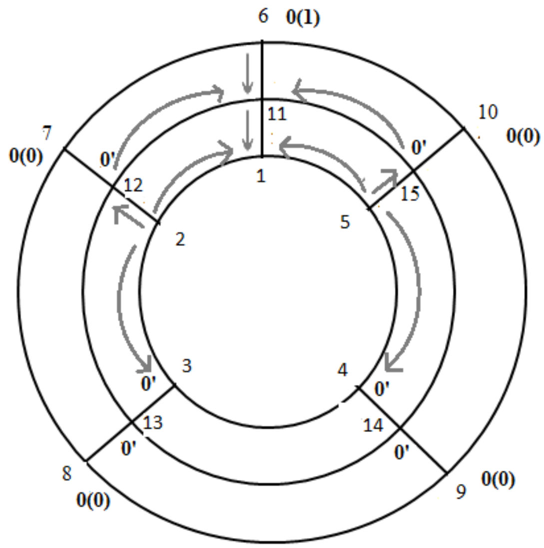


Figure 2.9: The contracted G(5,3) with the initial conditions in the outer-boundary vertices and the current flow.

$$\begin{bmatrix} \lambda_{6,1} & \lambda_{6,2} & \lambda_{6,3} & \lambda_{6,4} & \lambda_{6,5} \\ \lambda_{7,1} & \lambda_{7,2} & \lambda_{7,3} & \lambda_{7,4} & \lambda_{7,5} \\ \lambda_{8,1} & \lambda_{8,2} & \lambda_{8,3} & \lambda_{8,4} & \lambda_{8,5} \\ \lambda_{9,1} & \lambda_{9,2} & \lambda_{9,3} & \lambda_{9,4} & \lambda_{9,5} \\ \lambda_{10,1} & \lambda_{10,2} & \lambda_{10,3} & \lambda_{10,4} & \lambda_{10,5} \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Figure 2.10: The linear system of equations to compute the inner-boundary voltages, the voltage at vertex six is equal to zero.

$$\begin{bmatrix} \lambda_{1,1} & \lambda_{1,2} & \lambda_{1,3} & \lambda_{1,4} & \lambda_{1,5} & \lambda_{1,6} & \lambda_{1,7} & \lambda_{1,8} & \lambda_{1,9} & \lambda_{1,10} \\ \lambda_{2,1} & \lambda_{2,2} & \lambda_{2,3} & \lambda_{2,4} & \lambda_{2,5} & \lambda_{2,6} & \lambda_{2,7} & \lambda_{2,8} & \lambda_{2,9} & \lambda_{2,10} \\ \lambda_{3,1} & \lambda_{3,2} & \lambda_{3,3} & \lambda_{3,4} & \lambda_{3,5} & \lambda_{3,6} & \lambda_{3,7} & \lambda_{3,8} & \lambda_{3,9} & \lambda_{3,10} \\ \lambda_{4,1} & \lambda_{4,2} & \lambda_{4,3} & \lambda_{4,4} & \lambda_{4,5} & \lambda_{4,6} & \lambda_{4,7} & \lambda_{4,8} & \lambda_{4,9} & \lambda_{4,10} \\ \lambda_{5,1} & \lambda_{5,2} & \lambda_{5,3} & \lambda_{5,4} & \lambda_{5,5} & \lambda_{5,6} & \lambda_{5,7} & \lambda_{5,8} & \lambda_{5,9} & \lambda_{5,10} \\ \lambda_{6,1} & \lambda_{6,2} & \lambda_{6,3} & \lambda_{6,4} & \lambda_{6,5} & \lambda_{6,6} & \lambda_{6,7} & \lambda_{6,8} & \lambda_{6,9} & \lambda_{6,10} \\ \lambda_{7,1} & \lambda_{7,2} & \lambda_{7,3} & \lambda_{7,4} & \lambda_{7,5} & \lambda_{7,6} & \lambda_{7,7} & \lambda_{7,8} & \lambda_{7,9} & \lambda_{7,10} \\ \lambda_{8,1} & \lambda_{8,2} & \lambda_{8,3} & \lambda_{8,4} & \lambda_{8,5} & \lambda_{8,6} & \lambda_{8,7} & \lambda_{8,8} & \lambda_{8,9} & \lambda_{8,10} \\ \lambda_{9,1} & \lambda_{9,2} & \lambda_{9,3} & \lambda_{9,4} & \lambda_{9,5} & \lambda_{9,6} & \lambda_{9,7} & \lambda_{9,8} & \lambda_{9,9} & \lambda_{9,10} \\ \lambda_{10,1} & \lambda_{10,2} & \lambda_{10,3} & \lambda_{10,4} & \lambda_{10,5} & \lambda_{10,6} & \lambda_{10,7} & \lambda_{10,8} & \lambda_{10,9} & \lambda_{10,10} \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \\ u_7 \\ u_8 \\ u_9 \\ u_{10} \end{bmatrix} = \begin{bmatrix} i_1 \\ i_2 \\ i_3 \\ i_4 \\ i_5 \\ i_6 \\ i_7 \\ i_8 \\ i_9 \\ i_{10} \end{bmatrix}$$

Figure 2.11: The currents in the boundary vertices is given by the product of the response matrix Λ' and the boundary voltages.

$$\gamma_{2,3} = \frac{i_{2,3}}{u_2 - u_3}, \gamma_{5,4} = \frac{i_{5,4}}{u_5 - u_4} \tag{2.6}$$

Future Work:

For graphs $G(2n-1,n), n \in \mathbb{R}^{+}$, there appear to be some similarities with the graph $G(3,2)$ and $G(5,3)$. Moreover, if the properties of bigger graphs correspond to the conclusions obtained in this research then any $G(2n-1,n), n \in \mathbb{R}^{+}$ graph will be unrecoverable. On another note, the real-case scenario of this application is deriving a way for circuits to be analyzed; in more detail, if one of the conductors in a circuit is malfunctioning there should exist a way to find such conductor using the response matrix and initial conditions on the boundary nodes of the circuit being analyzed.

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